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GENERAL SOLUTION AND GREEN FUNCTION WITH BIFURCATION FOR NONLINEAR PLANE STRESS DEFORMATION OF A COMPRESSIBLE WEDGE

Z. YONG
Center for Robotics & Manufacturing Systems, University of Kentucky, Lexington, KY 40506-0108, U.S.A.

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Abstract—Exact stress, strain and displacement fields with closed form are determined for a nonlinear boundary value wedge problem. The compressible wedge experiencing small plane stress deformation is loaded by a concentrated force at its apex and the material is assumed to satisfy the power-law $\sigma_E = E_0\varepsilon_E^n$ where E_0 and n $(0 < n \leqslant 1)$ are positive constants, σ_E and ε_E are the stress and strain intensities, respectively. The results show that bifurcation with three branches occurs when the value of n is close to $\nu/(1+\nu)$ where ν is the Poisson ratio. The discontinuity of displacement components and their gradients proves to exist if the solution pertaining to one branch characterized by $n = \nu/(1+\nu)$ is required to satisfy the symmetric and θ-dependence conditions. These phenomena can be ascribed to the property conversion of the governing equation from the elliptic $(n > \nu/(1+\nu))$ to parabolic $(n = \nu/(1+\nu))$ or hyperbolic type $(n < \nu/(1+\nu))$. As illustrative examples, the Green functions are ascertained for a half plane subjected to a normal and a shear forces, respectively. It is found that the stress distribution for symmetric problems of the three branches fundamentally differs from each other. For deformation of the elliptic type, two fanlike tensile zones are discovered near the boundary of the half plane loaded by a compressive normal force. Copyright © 1996 Elsevier Science Ltd.

1. INTRODUCTION

From the point of view of engineering applications, the nonlinear boundary value problem of a plane compressible wedge subjected to a concentrated force at its apex is one of the most significant topics in the analysis of plane deformation. The general exact solution for this nonlinear problem can be used to analyze the bending of a fan-shaped beam, the contact stress between two plane bodies and the stress distribution of an orthogonal cutting tool during a machining process and so on. Additionally, the stress, strain and displacement fields with explicit form for a half plane, namely the Green function, can play a key role in formulating integral equations in the nonlinear boundary element method. To date, there seems to be no such analytical solution reported in the literature available to the author.

The nonlinear behavior of a compressible homogeneous isotropic material under concern stems from the classical power-law derived from Hencky's model (Kachanov, 1971). The rationale about the reasoning is that Hencky's model is essentially a nonlinear elastic one despite its application to limited plastic problems. Consequently, the form of constitutive equations in this work appears to be similar with the traditional Hooke's law except that the Young modulus E is no longer a constant. A constitutive model of this kind makes it relatively easier to achieve the analytical solution of a nonlinear elastic problem.

Another equally important role that an exact solution may play is to enhance the theoretical understanding about the complexity of nonlinear phenomena in solid mechanics, such as the bifurcation, discontinuity and hypersingularity of solutions. In this area, many important issues are well-illustrated and remarkable research work can be found in the literature (e.g., Keller and Antman, 1969; Hill and Hutchinson, 1975; Knowles, 1977; Abeyaratne and Knowles, 1987; Rajagopal and Carroll, 1992; Rajagopal and Tao, 1992; Jiang, 1994). Based on the bounded or unbounded structure of pressure field, in particular, the finite deformation of power law materials was extensively investigated within an incompressible wedge of plane strain by Rajagopal and Carroll (1992), Rajagopal and Tao

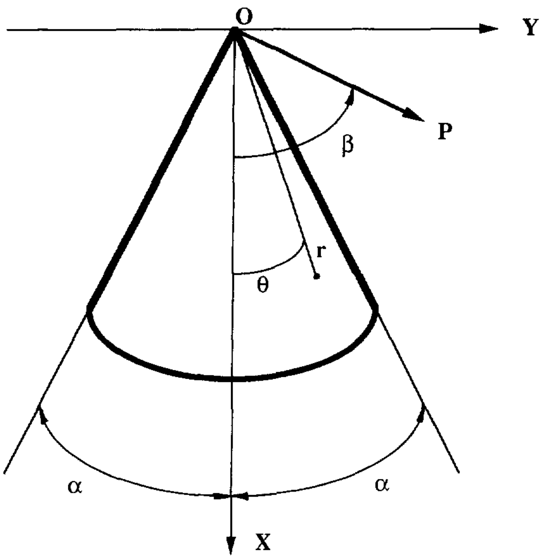

Fig. 1. A wedge subjected to a concentrated force at its apex.

(1992) and Fu *et al.* (1990), mainly regarding the possible distribution of homogeneous or inhomogeneous deformation of the wedge. Some of the interesting results discovered by them are much likely to be similar to several phenomena reported here, such as the distinct deformation zones near boundaries before and after the loss of ellipticity. In addition, the general behavior of bifurcation appearing in the present work has certain common features with that of the tensile deformation examined by Hill and Hutchinson (1975). To have an insight into the basic property of a power-law material, readers are invited to study the research work by Knowles (1977) considering both geometric and physical nonlinearities for plane crack and punch problems.

Attention of this research focuses on determination of exact solutions to nonlinear plane stress deformation of a wedge loaded by an arbitrarily-oriented force at its apex, as shown in Fig. 1. In the sequel, a group of nonlinear properties of the solutions is revealed, such as bifurcation, discontinuity and hypersingularity. Another investigation by the author (Yong, 1995) demonstrates that the occurrence of these nonlinear properties can be attributed to the transmission of the governing equation from the elliptic ($n > \nu/(1+\nu)$) to parabolic ($n = \nu/(1+\nu)$) or hyperbolic ($n < \nu/(1+\nu)$) type. In nonlinear plane strain deformation, these criteria are quite different (Yong, 1995). As an illustrative example of applications, the Green functions for a half plane are derived from three kinds of general solutions and they exhibit many interesting phenomena. For example, neighboring on a dominant compressive region, two unusual tensile zones near the boundary of the half plane loaded by a normal compressive force prove to exist for the elliptic-type material; this finding provides a theoretical basis for understanding the mechanism of appearance of surface cracks around the loading area in two dimensional contact problems. In addition, symmetric deformation patterns for elliptic, parabolic and hyperbolic materials differ greatly from each other.

2. FORMULATION

A cylindrical coordinate system (r, θ, z) will be used throughout the investigation. In the following solution procedure, all the basic assumptions about plane stress wedge deformation except the constitutive model are as same as those made in the linear analysis (Timoshenko and Goodier, 1970). That is, the only non-zero stress component is the radial stress σ_{rr} and non-zero strain components are three normal strain ε_{rr}, $\varepsilon_{\theta\theta}$ and ε_{zz} $(0 \leqslant r < \infty, -\alpha \leqslant \theta \leqslant \alpha)$. In such a circumstance, a boundary value problem as illustrated in Fig. 1 is formed by a group of simplified field equations and boundary conditions.

Equilibrium equation

Three equilibrium equations with no body force reduce to one, it is written as

$$\frac{\partial \sigma_{rr}}{\partial r}+\frac{\sigma_{rr}}{r}=0. \tag{1}$$

Compatible equation

It is well-known that there is only one compatible equation in plane problems. Since $\gamma_{r\theta}=0$, the equation has a simplified form

$$r^2\frac{\partial^2 \varepsilon_{\theta\theta}}{\partial r^2}+\frac{\partial^2 \varepsilon_{rr}}{\partial \theta^2}+2r\frac{\partial \varepsilon_{\theta\theta}}{\partial r}-r\frac{\partial \varepsilon_{rr}}{\partial r}=0. \tag{2}$$

Strictly speaking, eqn (2) exactly holds for plane strain deformation. For plane stress deformation, other slight influences could occur on the accuracy of compatibility. More information about this issue is available in the literature (Timoshenko and Goodier, 1970, p. 31).

Constitutive equation

The power-law model adopted by the present research is a modified form of the Hencky's relation in plasticity (Kachanov, 1971). Three basic assumptions attached with this model are discussed in detail below.

(i) The stress and strain deviatorics are proportional. The mathematical statement of this assumption is

$$\frac{e_{ij}}{s_{ij}}=\frac{1+\nu}{E}\quad (i,j=1,2,3), \tag{3}$$

where $s_{ij}=\sigma_{ij}-\sigma\delta_{ij}$ and $e_{ij}=\varepsilon_{ij}-\varepsilon\delta_{ij}$ are components of the stress and strain deviatorics, σ_{ij} and ε_{ij} are components of stress and strain tensors, $\sigma=\sigma_{ii}/3$ and $\varepsilon=\varepsilon_{ii}/3$ are the mean pressure and strain, δ_{ij} is the Kronecker symbol. For nonlinear elastic materials, the Young modulus E in eqn (3) is no longer a constant but the Poisson ratio ν $(0<\nu\leqslant 1/2)$ is still treated as a constant. After ordinary calculations, constitutive equations between σ_{ij} and ε_{ij} can be derived from eqn (3) in the form

$$\varepsilon_{ij}=\frac{1+\nu}{E}\left(\sigma_{ij}-\frac{3\nu}{1+\nu}\sigma\delta_{ij}\right), \tag{4}$$

where the relation

$$\varepsilon=\frac{1-2\nu}{E}\sigma \tag{5}$$

is employed in the derivation.

(ii) Stress intensity σ_E is a function of strain intensity ε_E, namely

$$\sigma_E=F(\varepsilon_E), \tag{6}$$

where σ_E and ε_E are defined by

$$\sigma_E = \sqrt{\tfrac{3}{2} s_{ij} s_{ij}}, \tag{7}$$

$$\varepsilon_E = \frac{1}{1+\nu}\sqrt{\tfrac{3}{2} e_{ij} e_{ij}}. \tag{8}$$

It is necessary to point out that σ_E and ε_E are in agreement with unidirectional tests. After substitution of (3) into (7), with the aid of (8) one can obtain

$$E = \frac{\sigma_E}{\varepsilon_E}. \tag{9}$$

Again, E matches with the physical meaning of a tensile or compressive test. For a power-law material, the specific expression of eqn (6) is written as

$$\sigma_E = E_0 \varepsilon_E^n, \tag{10}$$

where E_0 and n $(0 < n \leqslant 1)$ are physical constants of the material determined by tensile or compressive tests. As a result, E is converted to

$$E = E_0 \varepsilon_E^{n-1}. \tag{11}$$

This result will be applied to ascertain constitutive equations of the wedge problem.

(iii) The relation between the mean strain and pressure given by eqn (5) is nonlinearly elastic. In the Hencky's model, the same relation is assumed to be linearly elastic. Some previous experiments claimed that eqn (5) could approximately be a linear one, that is, the Young modulus was treated as a constant. However, quite a few arguments retain for experimental conditions of arbitrary loading. For example, in the unidirectional tension, the claim about the linear feature of eqn (5) breaks down in any nonlinear case. This is one significant difference between the present and Hencky's models.

Now attention is turned to the wedge problem. According to eqns (4, 8, 11), the constitutive equation in this case is formulated as

$$\varepsilon_{rr} = \frac{\sigma_{rr}}{E} = \frac{\sigma_{rr}}{E_0 |\varepsilon_{rr}|^{n-1}} = -\frac{\varepsilon_{\theta\theta}}{\nu} = -\frac{\varepsilon_{zz}}{\nu}. \tag{12}$$

Equation (12) will play an important role in determination of solutions for the boundary value problem.

Boundary condition

On the boundaries of the wedge, stress components must satisfy the conditions

$$\sigma_{\theta\theta}(r, \pm\alpha) = \sigma_{r\theta}(r, \pm\alpha) = 0 \quad 0 \leqslant \alpha < \pi, \tag{13}$$

$$P\cos\beta + \int_{-\alpha}^{\alpha} \sigma_{rr} r \cos\theta \, d\theta = 0, \tag{14}$$

$$P\sin\beta + \int_{-\alpha}^{\alpha} \sigma_{rr} r \sin\theta \, d\theta = 0. \tag{15}$$

Note that boundary condition (13) automatically holds since $\sigma_{r\theta} = \sigma_{\theta\theta} = 0$ is an identity.

The strategy for determining solutions of the nonlinear wedge problem is outlined as follows.

(a) Solve compatible eqn (2) and then determine appropriate forms of strain components ($\varepsilon_{rr}, \varepsilon_{\theta\theta}, \varepsilon_{zz}$) which are physically reasonable.

(b) Find stress component σ_{rr} with a free unknown function by use of equilibrium equation (1).

(c) Connect σ_{rr} to ($\varepsilon_{rr}, \varepsilon_{\theta\theta}, \varepsilon_{zz}$) by means of constitutive eqn (12) and ascertain the final form of the free function contained by σ_{rr}. As a result, σ_{rr} satisfies both equilibrium and compatible equations.

After strain components ε_{ij} are known, the geometric equations

$$\varepsilon_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) \tag{16}$$

are used to determine displacement components u_i. Moreover, certain unknown constants can be evaluated by boundary conditions (14, 15).

To concentrate on the discussion of significant nonlinear properties that solutions possess, the detailed solving process is assigned to the Appendix. Following such an arrangement, exact stress, strain and displacement components are directly listed in the following.

(a) Stress component

$$\sigma_{rr} = E_0 \operatorname{sign}(\Theta)|\Theta|^n r^{-1}. \tag{17}$$

The function sign(f) is defined by

$$\operatorname{sign}(f) = \frac{f}{|f|} = \begin{cases} 1 & f > 0, \\ -1 & f < 0 \end{cases}, \tag{18}$$

where f is a function or a variable.

(b) Strain components

$$\varepsilon_{rr} = \Theta r^{-1/n}, \tag{19}$$

$$\varepsilon_{zz} = \varepsilon_{\theta\theta} = -\nu\varepsilon_{rr}. \tag{20}$$

(c) Displacement components

$$u_r = \frac{n}{n-1}\Theta r^{(n-1)/n} + c_3 \cos\theta + c_4 \sin\theta \quad (n \neq 1), \tag{21}$$

$$u_\theta = -\left(\frac{n}{n-1} + \nu\right)\bar{\Theta} r^{(n-1)/n} - c_3 \sin\theta + c_4 \cos\theta + c_5 n r^{(n-1)/n} + c_6 r + c_7 \quad (n \neq 1). \tag{22}$$

Here c_i ($i = 3-7$) are constants. Displacement components for the linear case ($n = 1$) are not displayed here for the sake of compactness. Functions Θ and $\bar{\Theta}$ in above expressions are categorized by the relation between the Poisson ratio ν and exponent n and they are illustrated below.

(i) $n < \nu/(1+\nu)$:

$$\Theta = c_1 \operatorname{ch}(\lambda_a\theta) + c_2 \operatorname{sh}(\lambda_a\theta), \tag{23}$$

$$\bar{\Theta} = \frac{c_1}{\lambda_a}\operatorname{sh}(\lambda_a\theta) + \frac{c_2}{\lambda_a}\operatorname{ch}(\lambda_a\theta), \tag{24}$$

$$\lambda_a = \frac{1}{n}\sqrt{(1+\nu)\left(\frac{\nu}{1+\nu}-n\right)}. \tag{25}$$

(ii) $n = \nu/(1+\nu)$:

$$\Theta = c_1 + c_2\theta, \quad \bar{\Theta} = c_1\theta + c_2\frac{\theta^2}{2}; \tag{26}$$

$$\Theta = c_1 + c_2|\theta|, \quad \bar{\Theta} = c_1\theta + c_2\,\mathrm{sign}(\theta)\frac{\theta^2}{2}; \tag{27}$$

$$c_5 = \frac{n}{n-1}c_2 \quad \text{when } \Theta = c_1 + c_2\theta; \tag{28}$$

$$c_5 = \frac{n}{n-1}\mathrm{sign}(\theta)c_2 \quad \text{when } \Theta = c_1 + c_2|\theta|. \tag{29}$$

(iii) $n > \nu/(1+\nu)$:

$$\Theta = c_1\cos(\lambda_b\theta) + c_2\sin(\lambda_b\theta), \tag{30}$$

$$\bar{\Theta} = \frac{c_1}{\lambda_b}\sin(\lambda_b\theta) - \frac{c_2}{\lambda_b}\cos(\lambda_b\theta), \tag{31}$$

$$\lambda_b = \frac{1}{n}\sqrt{(1+\nu)\left(n-\frac{\nu}{1+\nu}\right)}. \tag{32}$$

According to eqns (30, 32), as a special case, σ_{rr} and ε_{rr} are in agreement with linear results when $n = 1$. Usually, constants c_1 and c_2 in above equations can be calculated by employing boundary conditions (14, 15), and values of the other constants depend on boundary conditions of displacement components and/or their gradients. It will be seen that an additional condition is needed if eqn (27) is used to analyze a symmetric problem. After substitution of eqn (17) into (14) and (15), the two boundary conditions are converted to

$$P\cos\beta + E_0\int_{-\alpha}^{\alpha}\frac{\Theta}{|\Theta|}|\Theta|^n\cos\theta\,\mathrm{d}\theta = 0, \tag{33}$$

$$P\sin\beta + E_0\int_{-\alpha}^{\alpha}\frac{\Theta}{|\Theta|}|\Theta|^n\sin\theta\,\mathrm{d}\theta = 0. \tag{34}$$

Obviously, bifurcation with three branches takes place when n is close to $\nu/(1+\nu)$. Another analysis (Yong, 1995) shows that in plane strain deformation, bifurcation also occurs but its bifurcation point is positioned by $n = \nu$ instead of $n = \nu/(1+\nu)$. The conclusion can be drawn from common calculations that the low limit of $n = \nu/(1+\nu)$ is significantly smaller than that of $n = \nu$. For instance, when $\nu = 1/3$ one obtains $n = 1/4$ for plane stress and $n = 1/3$ for plane strain. The smaller limit suggests that in plane stress the ellipticity of governing equations varies within larger scope in comparison with the situation of plane strain. Owing to bifurcation, a structure with material property $n \approx \nu/(1+\nu)$ could be unstable because any small disturbance around the bifurcation point may lead to an uncertain deformation pattern.

It turns out from eqns (27) and (29) that the circumferential displacement, displacement gradients, strain and stress components contain discontinuity at the symmetric plane $\theta = 0$ of the wedge if $n = v/(1+v)$. It will be demonstrated that this type of discontinuity is inevitable in order to ensure the θ-dependence of the strain–stress field for a general symmetric problem. As a matter of fact, it is one of the most interesting phenomena caused by bifurcation or loss of ellipticity, the latter topic has been extensively examined from different points of view (e.g., Knowles, 1977; Abeyaratne and Knowles, 1987).

Another important issue resulting from bifurcation or loss of ellipticity is the hyper-singularity of strain or displacement components. That is, $-1/n \to -\infty$ or $(n-1)/n \to -\infty$ in eqns (19–22) when $n \to 0$. The physical meaning of the theoretical reasoning is that very high strain concentration may exist around the loading area for the branch of $n < v/(1+v)$.

As discussed above, the only source of nonlinearity in this work comes from the classical constitutive eqn (11), and the geometric relation is a linear one as shown in eqn (16). These governing equations may be challenged by the rational requirement of the frame indifference (Bowen, 1989) for large deformation, but for problems of small deformation, their effectiveness has long been accepted by engineers and researchers. In addition, if the off-loading path is assumed not to match the loading path, the material response as a whole is no longer of the elastic feature.

3. APPLICATION

As an application of general solutions, in this section the Green functions are analyzed for a half plane loaded by a shear and a normal concentrated force, respectively. In the two cases, geometric parameters in Fig. 1 are identified by $\alpha = \pi/2$, $\beta = 0$ and $\alpha = \pi/2$, $\beta = \pi/2$.

Since $\alpha = \pi/2$ and $\beta = 0$ are relevant to a symmetric case and $\alpha = \beta = \pi/2$ to an antisymmetric one, constant c_1 or c_2 in function Θ should be equal to zero, except for the case of $\Theta = c_1 + c_2|\theta|$, so that Θ can be an odd or an even function. Additionally, one of the boundary conditions (33, 34) automatically vanishes because the integration of an odd function is equal to zero when it is associated with symmetric integral limits.

In the branch of $n > v/(1+v)$, the value of $\lambda_b = \sqrt{(1+v)[n-v/(1+v)]/n}$ has very significant influence on the classification of tensile and compressive regions, thus accurate evaluation on it is necessary. First, one can readily verify that $\lambda_b \geqslant 1$ holds if

$$v \leqslant n \quad (0 < n < 1). \tag{35}$$

Furthermore, assume that λ_b is a function of n and then in light of the equation

$$\frac{d\lambda_b}{dn} = \frac{1+v}{2n^2\sqrt{n(1+v)-v}}\left(\frac{2v}{1+v} - n\right), \tag{36}$$

one knows that λ_b reaches its maximum value at $n = 2v/(1+v)$

$$(\lambda_b)_{max} = \lambda_{bm} = \frac{1+v}{2\sqrt{v}}. \tag{37}$$

In addition, it is easy to show that λ_{bm} given by eqn (37) is a monotonically decreasing function of v. For instance, when v increases from 1/5 to 1/2, λ_{bm} decreases from $3\sqrt{5}/5$ to $3\sqrt{2}/4$.

In the following discussion, deformation of a half plane is categorized into three types i.e., $n > v/(1+v)$, $n = v/(1+v)$ and $n < v/(1+v)$ as formulated in the Section 2. Both shear and normal point loadings are considered in each category.

1(a) $n > v/(1+v)$, $\alpha = \pi/2$ and $\beta = 0$. This is a symmetric problem and hence c_2 is taken as zero. Substituting $\Theta = c_1 \cos(\lambda_b\theta)$ into eqn (33) leads to

$$P+2E_0\int_0^{\pi/2}\frac{c_1\cos(\lambda_b\theta)}{|c_1\cos(\lambda_b\theta)|}|c_1\cos(\lambda_b\theta)|^n\cos\theta\,\mathrm{d}\theta = 0. \tag{38}$$

If $\lambda_b \leqslant 1$ then $c_1(c_1 < 0)$ can be written in the form

$$c_1 = -\left\{\frac{P}{2E_0\int_0^{\pi/2}[\cos(\lambda_b\theta)]^n\cos\theta\,\mathrm{d}\theta}\right\}^{1/n}. \tag{39}$$

When $\lambda_b \geqslant 1$ eqn (38) is rearranged to

$$P+2E_0\frac{c_1}{|c_1|}|c_1|^n I_\lambda = 0, \tag{40}$$

where

$$I_\lambda = \int_0^{\pi/(2\lambda_b)}[\cos(\lambda_b\theta)]^n\cos\theta\,\mathrm{d}\theta-\int_{\pi/(2\lambda_b)}^{\pi/2}|\cos(\lambda_b\theta)|^n\cos\theta\,\mathrm{d}\theta. \tag{41}$$

Numerical analysis shows that $I_\lambda > 0$ holds for $\lambda_b = 3\sqrt{2}/4 \sim 3\sqrt{5}/5$ as confined by eqn (37). As a result, $c_1 < 0$ is confirmed and c_1 is given by

$$c_1 = -\left(\frac{P}{2E_0I_\lambda}\right)^{1/n}. \tag{42}$$

Note that eqn (42) reduces to eqn (39) when $\lambda_b = 1$ in the integral limits of (41), thus the deformation analysis relevant to (39) is not singled out as an independent part. By means of eqns (17, 19), the stress and strain components for $\lambda_b \geqslant 1$ are written in the forms

$$\sigma_{rr} = -\frac{P}{2I_\lambda}[\cos(\lambda_b\theta)]^n r^{-1} \quad |\theta| \leqslant \frac{\pi}{2\lambda_b}, \tag{43}$$

$$\sigma_{rr} = \frac{P}{2I_\lambda}|\cos(\lambda_b\theta)|^n r^{-1} \quad \frac{\pi}{2\lambda_b} \leqslant |\theta| \leqslant \pi/2; \tag{44}$$

$$\varepsilon_{rr} = -\left(\frac{P}{2E_0I_\lambda}\right)^{1/n} r^{-1/n}\cos(\lambda_b\theta) \quad |\theta| \leqslant \pi/2. \tag{45}$$

Tensile and compressive regions of stress and strain components are consistent with each other. Based on symmetric features of the problem, conclusion c_j ($j = 4 \sim 7$) $= 0$ for eqns (21, 22) can be reached because displacement components u_r and u_θ are an even and odd function, respectively. From general expression (21, 22), u_r and u_θ are of the forms

$$u_r = -\frac{n}{n-1}\left(\frac{P}{2E_0I_\lambda}\right)^{1/n} r^{(n-1)/n}\cos(\lambda_b\theta)+c_3\cos\theta \quad (n \neq 1), \tag{46}$$

$$u_\theta = \frac{1}{\lambda_b}\left(\frac{n}{n-1}+v\right)\left(\frac{P}{2E_0I_\lambda}\right)^{1/n} r^{(n-1)/n}\sin(\lambda_b\theta)-c_3\sin\theta \quad (n \neq 1). \tag{47}$$

The value of constant c_3 depends on the geometric constraint of a specific problem and

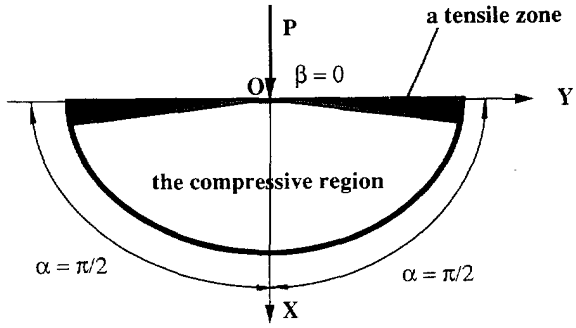

Fig. 2. The illustration of tensile zones neighboring on the dominant compressive region.

hence it is not discussed here for brevity. As one may notice, the most interesting phenomenon reflected from eqns (43–45) is that neighboring on the dominant compressive region there exist two symmetric tensile zones in the vicinity of boundary of a half plane, as shown in Fig. 2. The similar result is also obtained for plane strain deformation but the angle covering a zone in plane stress is larger than the angle in plane strain. As displayed in eqn (37), the maximum value of λ_b is controlled by the minimum value of the Poisson ratio, therefore the tensile zones cannot become dominant within a half plane. During the contact between two plane bodies, the tensile stress in these zones may be the major driving force to generate surface cracks around the contact region.

1(b) $n > v/(1+v)$, $\alpha = \beta = \pi/2$. This is an antisymmetric case. The appropriate form of Θ is $\Theta = c_2 \sin(\lambda_b\theta)$ and naturally c_1 is taken as zero. Inserting $\Theta = c_2\sin(\lambda_b\theta)$ into the boundary condition (34) yields

$$P+2E_0\int_0^{\pi/2}\frac{c_2\sin(\lambda_b\theta)}{|c_2\sin(\lambda_b\theta)|}|c_2\sin(\lambda_b\theta)|^n\sin\theta\,d\theta = 0. \tag{48}$$

Since $\sin(\lambda_b\theta) \geqslant 0$ for $0 \leqslant \lambda_b \leqslant 2$ within $[0, \pi/2]$, the expression for $c_2 (c_2 < 0)$ is of the form

$$c_2 = -\left\{\frac{P}{2E_0\displaystyle\int_0^{\pi/2}[\sin(\lambda_b\theta)]^n\sin\theta\,d\theta}\right\}^{1/n}. \tag{49}$$

Based on eqn (17), σ_{rr} is given by

$$\sigma_{rr} = -\operatorname{sign}(\theta)\frac{P}{2\displaystyle\int_0^{\pi/2}[\sin(\lambda_b)]^n\sin\theta\,d\theta}|\sin(\lambda_b\theta)|^n r^{-1}. \tag{50}$$

Equation (50) is consistent in a common sense with the physical meaning, that is, $\sigma_{rr} \leqslant 0$ for $0 \leqslant \theta \leqslant \pi/2$ and $\sigma_{rr} \geqslant 0$ for $-\pi/2 \leqslant \theta \leqslant 0$. The analysis for strain and displacement components is similar to what one has implemented in the case 1(a) and thus it is omitted to avoid repetition.

2(a) $n = v/(1+v)$, $\alpha = \pi/2$ and $\beta = 0$. The solution for the branch is classified as the parabolic type. A very interesting issue arising here is how to chose an appropriate form for function Θ. As shown in eqns (26, 27), to be an even function, Θ only has two reasonable choices. One is $\Theta = c_1$ and the other is $\Theta = c_1 + c_2|\theta|$. The probability of $\Theta = c|\theta|$ is excluded because the result $\sigma_{rr} = 0$ at $\theta = 0$ is most unlikely to match with the practical meaning despite the existence of a mathematical solution.

As indicated before, boundary condition (34) automatically disappears due to the constraint of symmetric integration limits on an odd function, and therefore eqn (33) is the only condition for calculating constants. From this point of view, the choice $\Theta = c_1$ is seemingly a suitable one. Nevertheless, the stress and strain components derived from this choice are independent of polar angle θ at all. The result of this kind may occasionally have its own physical background but it is, in most respects, far beyond the traditional belief about the distribution of stress and strain in a solid body especially for a general symmetric problem.

Based on the foregoing analysis, consequently, the expression $\Theta = c_1 + c_2|\theta|$ is a more reasonable form since it reflects the influence of θ. However, eqn (33) alone is not sufficient to determine two constants c_1 and c_2. Therefore, one more additional condition is necessary. As an illustrative example, the assumption $\sigma_{rr}(r, \pm\pi/2) = 0$, which may not be necessarily an exact one for the general situation, is introduced here in order to simplify the analytical procedure. More sophisticated discussions into this issue are left to future study. The equation given by

$$\Theta = c_1\left(1 - \frac{2}{\pi}|\theta|\right) \quad \left(c_2 = -\frac{2}{\pi}c_1\right) \tag{51}$$

meets the condition $\sigma_{rr}(r, \pm\pi/2) = 0$. After substitution of eqn (51) into (33), certain calculations leads to

$$c_1 = -\left[\frac{P}{2E_0\displaystyle\int_0^{\pi/2}(1-2|\theta|/\pi)^n\cos\theta\,\mathrm{d}\theta}\right]^{1/n}. \tag{52}$$

The radial stress σ_{rr} is of the form

$$\sigma_{rr} = -\frac{P}{2\displaystyle\int_0^{\pi/2}(1-2|\theta|/\pi)^n\cos\theta\,\mathrm{d}\theta}\left(1-\frac{2}{\pi}|\theta|\right)^n r^{-1} \tag{53}$$

which is in agreement with the conventional meaning. One intriguing phenomenon that appears in this case is discontinuity of the circumferential displacement, displacement gradients, stress and strain components at $\theta = 0$. The issue arises because u_θ contains the discontinuous terms $c_5 n r^{(n-1)/n}$ where $c_5 = n\,\mathrm{sign}(\theta)c_2/(n-1)$, and all the gradients are coherent to discontinuity of the first derivative of $|\theta|$. At $\theta = 0$ function $\mathrm{sign}(\theta)$ is defined as zero for u_θ which is an odd function with the form

$$u_\theta = -\left(\frac{n}{n-1}+v\right)c_1\left[\theta - \frac{1}{\pi}\mathrm{sign}(\theta)\theta^2\right]r^{(n-1)/n} - c_3\sin\theta - \frac{2n^2c_1}{\pi(n-1)}\mathrm{sign}(\theta)r^{(n-1)/n}. \tag{54}$$

Discontinuity of this kind is caused by the θ-dependence of stress and strain components, it may differ from the discontinuity resulting from the transmission of material properties. The latter one is meticulously examined in another part of this research (Yong, 1995). For more information about this topic, readers refer to other references (e.g., Abeyaratne and Knowles, 1987).

2(b) $n = v/(1+v)$, $\alpha = \beta = \pi/2$. To satisfy the requirement of antisymmetry, $\Theta = c_2\theta$ is a proper expression. Careful analysis into eqn (34) results in

$$c_2 = -\left[\frac{P}{2E_0 \int_0^{\pi/2} \theta^n \sin\theta \, d\theta}\right]^{1/n} \tag{55}$$

and then one can find from eqn (17)

$$\sigma_{rr} = -\text{sign}(\theta)\frac{P}{2\int_0^{\pi/2} \theta^n \sin\theta \, d\theta}|\theta|^n r^{-1}. \tag{56}$$

The stress distribution given by eqn (56) is coincident with the physical meaning in a common sense. For instance, $|\sigma_{rr}|$ reaches its maximum value when $\theta = \pm\pi/2$ and its minimum when $\theta = 0$.

3(a) $n < v/(1+v)$, $\alpha = \pi/2$ and $\beta = 0$. As mentioned previously the governing equation in this branch belongs to the hyperbolic type. It will be seen that the structure of solutions is dramatically changed due to the transmission of features of equations. Substitution of $\Theta = c_1 \text{ch}(\lambda_a\theta)$ into eqn (33) produces

$$c_1 = -\left[\frac{P}{2E_0 \int_0^{\pi/2} [\text{ch}(\lambda_a\theta)]^n \cos\theta \, d\theta}\right]^{1/n}. \tag{57}$$

By virtue of eqn (17), one obtains

$$\sigma_{rr} = -\frac{P}{2\int_0^{\pi/2} [\text{ch}(\lambda_a\theta)]^n \cos\theta \, d\theta}[\text{ch}(\lambda_a\theta)]^n r^{-1}. \tag{58}$$

Apparently, eqn (58) shows that all the half plane is in the compressive state and the situation seems to be as usual as the one described by eqn (43) for $\lambda_b \leqslant 1$ which contains no tensile zone. However, careful examinations indicate that there appears an unusual phenomenon in the stress distribution.

In the branch of $n > v/(1+v)$ if $\lambda_b \leqslant 1$ then the radial stress $|\sigma_{rr}|$ given by eqn (43) reaches its maximum value at $\theta = 0$ and its minimum value at or near $\theta = \pm\pi/2$, which means that the central part of a half plane burdens the majority of external loading. According to eqn (58), on the contrary, for the symmetric case of $n < v/(1+v)$ the major part of loading is supported by two lateral parts near the boundary because $|\sigma_{rr}|$ reaches its maximum at $\theta = \pm\pi/2$ and minimum value at $\theta = 0$. It follows that the conversion of physical properties of a material brings about fundamental changes of stress distribution.

3(b) $n < v/(1+v)$, $\alpha = \beta = \pi/2$. The appropriate form of Θ is $\Theta = c_2 \text{sh}(\lambda_b\theta)$. From eqns (34) and (17), one can obtain

$$c_2 = -\left[\frac{P}{2E_0 \int_0^{\pi/2} [\text{sh}(\lambda_a\theta)]^n \sin\theta \, d\theta}\right]^{1/n} \tag{59}$$

and

$$\sigma_{rr} = -\text{sign}(\theta)\frac{P}{2\int_0^{\pi/2}[\text{sh}(\lambda_a\theta)]^n \sin\theta \, d\theta}|\text{sh}(\lambda_a\theta)|^n r^{-1}. \tag{60}$$

Unlike the symmetric case, the general property of the radial stress for the antisymmetric case of $n < \nu/(1+\nu)$ does not alter significantly away from eqn (50) for $n > \nu/(1+\nu)$ and eqn (56) for $n = \nu/(1+\nu)$ despite the transmission of features of the governing equation. As a matter of fact, most fundamental changes of stress distribution take place in symmetric problems.

4. SUMMARY

Bifurcation is a primary characteristic of the nonlinear boundary value wedge problem. In each branch of the bifurcation, the solution exhibits its own special features. To have a clear pattern about these nonlinear phenomena, a brief summary is given as follows.

- The general solution contains bifurcation with three branches that emanate from $n = \nu/(1+\nu)$. The phenomenon can be ascribed to transmission of the governing equation from the elliptic to parabolic or hyperbolic type (Yong, 1995).
- Strain and displacement components in the branch of $n \leqslant \nu/(1+\nu)$ possess hyper-singularity at $r \to 0$ when $n \to 0$ which implies that there exists very high strain concentration around the loading area.
- In a symmetric case with $n = \nu/(1+\nu)$, the circumferential displacement, displacement gradients, stress and strain components contain discontinuity and it is unavoidable unless the θ-independence of stress and strain fields can be verified for a general symmetric problem.
- In the branch of $n > \nu/(1+\nu)$, there are fanlike tensile zones in the vicinity of boundary of a half plane loaded by a compressive normal force. This finding provides a rational explanation to the occurrence of surface cracks during the contact between two plane bodies. It is worthwhile to point out that the solution to a linear case ($n = 1$) does not give any indication about these kinds of tensile zones.
- The stress distribution for $n > \nu/(1+\nu)$ in a half plane with a normal point force differs fundamentally from the result for $n < \nu/(1+\nu)$. This suggests that the conversion of features of governing equations can cause great changes to the deformation pattern of a solid body.

Nonlinearity is the intrinsic property of materials. Compared with the linear theory, exact solutions are very rare for nonlinear boundary value problems due to the lack of well-developed methods like integral transforms. The situation provides an opportunity to researchers to explore new approaches in this area. Undoubtedly, it is worth making effects to search exact solutions since they may bring new significant information to engineers and researchers.

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APPENDIX

The traditional methods may not be effective for the nonlinear problem under consideration. A new approach is introduced in the following which greatly alleviates the difficulties in the solving process.

In light of eqn (12), substituting $\varepsilon_{\theta\theta} = -\nu\varepsilon_{rr}$ into compatible eqn (2) leads to the governing equation for ε_{rr}

$$r^2\frac{\partial^2\varepsilon_{rr}}{\partial r^2} - \frac{1}{\nu}\frac{\partial^2\varepsilon_{rr}}{\partial\theta^2} + \frac{2\nu+1}{\nu}r\frac{\partial\varepsilon_r}{\partial r} = 0. \tag{a-1}$$

Let

$$\varepsilon_{rr} = R(r)\Theta(\theta) \tag{a-2}$$

and then eqn (a-1) becomes

$$\left(r^2\frac{\mathrm{d}^2R}{\mathrm{d}r^2} + \frac{2\nu+1}{\nu}r\frac{\mathrm{d}R}{\mathrm{d}r}\right)\Theta - \frac{R}{\nu}\frac{\mathrm{d}^2\Theta}{\mathrm{d}\theta^2} = 0. \tag{a-3}$$

Equation (a-3) generates two ordinary differential equations

$$\frac{\mathrm{d}^2\Theta}{\mathrm{d}\theta^2} + \lambda\Theta = 0, \tag{a-4}$$

$$r^2\frac{\mathrm{d}^2R}{\mathrm{d}r^2} + \frac{2\nu+1}{\nu}r\frac{\mathrm{d}R}{\mathrm{d}r} + \frac{\lambda}{\nu}R = 0, \tag{a-5}$$

where λ is an unknown eigenvalue. The general solution to eqn (a-4) can readily be found in the forms

$$\Theta = c_1\,\mathrm{ch}(\sqrt{-\lambda}\theta) + c_2\,\mathrm{sh}(\sqrt{-\lambda}\theta) \quad \lambda < 0; \tag{a-6}$$

$$\Theta = c_1 + c_2\theta \quad \text{or} \quad \Theta = c_1 + c_2|\theta| \quad \lambda = 0; \tag{a-7}$$

$$\Theta = c_1\cos(\sqrt{\lambda}\theta) + c_2\sin(\sqrt{\lambda}\theta) \quad \lambda > 0, \tag{a-8}$$

where c_1 and c_2 are constants. It is necessary to point out that the function $\Theta = c_1 + c_2|\theta|$ satisfies eqn (a-4) and it is continuous at $\theta = 0$. Equation (a-5) is a Euler-type ordinary differential equation, its solution is given after usual calculations by

$$R = C_1 r^{-\eta_1} + C_2 r^{-\eta_2}, \tag{a-9}$$

$$\eta_{1,2} = \frac{1+\nu}{2\nu}\left[1 \pm \sqrt{1 - \frac{4\lambda\nu}{(1+\nu)^2}}\right] \quad 1 - \frac{4\lambda\nu}{(1+\nu)^2} > 0; \tag{a-10}$$

$$R = r^{-(1-\nu)/(2\nu)}(C_1 + C_2\ln r) \quad 1 - \frac{4\lambda\nu}{(1+\nu)^2} = 0; \tag{a-11}$$

$$R = r^{-(1+\nu)/(2\nu)}\left\{C_1\cos\left[\frac{1+\nu}{2\nu}\ln r\sqrt{\frac{4\lambda\nu}{(1+\nu)^2} - 1}\right] + C_2\sin\left[\frac{1+\nu}{2\nu}\ln r\sqrt{\frac{4\lambda\nu}{(1+\nu)^2} - 1}\right]\right\}$$

$$1 - \frac{4\lambda\nu}{(1+\nu)^2} < 0. \tag{a-12}$$

In the above equations, C_1 and C_2 are constants. Because $\varepsilon_{rr} \to 0$ must hold for $r \to \infty$ and ε_{rr} should be in agreement with the linear result when $n = 1$, the appropriate form of ε_{rr} is

$$\varepsilon_{rr} = r^{-\eta}\Theta, \tag{a-13}$$

where $\eta > 0$ is a real constant. It is easy to verify that the radial stress given by

$$\sigma_{rr} = \Theta_\sigma(\theta)r^{-1} \tag{a-14}$$

satisfies equilibrium eqn (1). Note that Θ_σ is an unknown free function, and it will be chosen in such a way that σ_{rr} also satisfies compatible eqn (2). Substitution of eqns (a-13) and (a-14) into (12) results in

$$\Theta_\sigma r^{-1} = E_0 \operatorname{sign}(\Theta)|r^{-\eta}\Theta|^n \tag{a-15}$$

and therefore one obtains

$$\eta = \frac{1}{n} \tag{a-16}$$

$$\Theta_\sigma = E_0 \operatorname{sign}(\theta)|\Theta|^n. \tag{a-17}$$

After replacement of $\eta_{1,2}$ in eqn (a-10) by $\eta_{1,2} = 1/n$, certain calculations result in the expression for the eigenvalue λ with the form

$$\lambda = \frac{1+v}{n^2}\left(n - \frac{v}{1+v}\right). \tag{a-18}$$

At this stage, eqns (a-6, 7, 8) are completely determined and they are in consistence with eqns (23–32).

Now attention is turned to evaluate the displacement field. In terms of $\varepsilon_{rr} = \Theta r^{-1/n} = \partial u_r/\partial r$, u_r is ascertained in the form

$$u_r = \frac{n}{n-1} r^{(n-1)/n}\Theta + \Phi, \tag{a-19}$$

where $\Phi = \Phi(\theta)$ is an unknown function of θ. In addition, application of the geometric equation

$$\varepsilon_{\theta\theta} = -v\varepsilon_{rr} = \frac{1}{r}\frac{\partial u_\theta}{\partial \theta} + \frac{u_r}{r} \tag{a-20}$$

yields the expression for u_θ

$$u_\theta = -\left(\frac{n}{n+1} + v\right) r^{(n-1)/n} \int \Theta \, d\theta - \int \Phi \, d\theta + R_u, \tag{a-21}$$

where $R_u = R_u(r)$ is an unknown function of r. To find the specific forms of R_u and Θ, substituting eqns (a-19, a-21) into

$$\varepsilon_{r\theta} = \frac{1}{r}\frac{\partial u_r}{\partial \theta} + \frac{\partial u_\theta}{\partial r} - \frac{u_\theta}{r} = 0, \tag{a-22}$$

one can obtain

$$\Phi = c_3 \cos\theta + c_4 \sin\theta, \tag{a-23}$$

$$R_u = c_5 n r^{(n-1)/n} + c_6 r + c_7, \tag{a-24}$$

where the constant c_5 is related to the equation

$$\frac{n-1}{n} c_5 = \frac{d\Theta}{d\theta} + \frac{1+v}{n^2}\left(n - \frac{v}{1+v}\right)\int \Theta \, d\theta. \tag{a-25}$$

Combining the preceding results, one can obtain the displacement components as shown in eqns (21, 22).